Table 2 - Parameters of the quadratic approximation of log $\overset{0}{e}_x$ for a few model life tables for the males.

Mortality level (1)	a (2)	$b\times10^2$ (3)	$c\times10^3$ (4)
10	1.5984	-.433	-.102
25	1.6467	-.313	-.119
40	1.6922	-.251	-.127
55	1.7375	-.253	-.128
70	1.7644	-.194	-.134
85	1.7942	-.193	-.134
100	1.8233	-.223	-.130

The estimated values of c are sufficiently small as these were expected to be. The values are negative, show a slowly declining trend and a sign of increase at the end. b, which is also small and negative, appears to be an oscillating but diminishing function of the levels. The increasing trend of 'a' is, however, more regular. Obviously, the estimates of these parameters, being dependent upon the model life tables, are subjected to the defects of those tables and hence should not be regarded as final. For a thorough analysis the parameters may be allowed to vary within tolerable limits and the results may be checked for consistency by comparisons with other life table functions. So far as the model life tables are concerned, the life expectancies reproduced from the estimated parameters compare favorably with the actual values. (Appendix 1)

3. Derivation of other life table functions

It is apparent from the results presented earlier that an alternative method for constructing model life tables can be formulated if it can be shown that the information about $\overset{0}{e}_x$ is sufficient to generate other life table functions. Fortunately, T_x can be determined directly from $\overset{0}{e}_x$, when l_x can be obtained from $l_x = T_x/\overset{0}{e}_x$ and hence the entire life table can be completed. This is so because

$$\int_{x_1}^{x_2} \frac{dx}{\overset{0}{e}_x} = \int_{x_1}^{x_2} \frac{l_x dx}{T_x} = \log_e (T_{x_1}/T_{x_2}) \quad (8)$$

Since $T_0 = l_0 \overset{0}{e}_0$ is known for a given model table, any T_x can be solved from (8) by putting $x_1 = 0$ and $x_2 = x$ provided the integral on the left hand side of (8) can be evaluated.

The values of $\overset{0}{e}_x$ can, however, be used to solve the above equation by numerical integration. It has been found that the approximation by trapezoidal rule even for five year intervals beyond age 5 is quite satisfactory. Thus

for $x_1 \geq 5$ and $x_2 - x_1 = 5$,

$$\log (T_{x_1} / T_{x_2}) =$$

$$\frac{1}{2} (x_2 - x_1) (1/\overset{0}{e}_{x_1} + 1/\overset{0}{e}_{x_2}) \quad (9)$$

For the first age interval (0,5), in which $\overset{0}{e}_x$ generally assumes its maximum value at $\hat{x}$, the recommended procedure is to use a quadratic approximation of $1/\overset{0}{e}_x$ for each of the two subintervals $(0, \hat{x})$ and $(\hat{x},5)$, with or even without any correction for equalizing the derivatives at $\hat{x}$. In each of these two cases, the quadratic is assumed to produce a minimum value at $\hat{x}$, thereby reducing the number of parameters from three to two. The parameters can therefore be estimated from the given values of $\overset{0}{e}_x$ at the two boundaries. Thus writing the quadratic as

$$1/\overset{0}{e}_x = m + nx + px^2 \quad (10)$$

and subjecting the equation to the condition that the minimum value is assumed at $\hat{x}$, the requirement for which is

$$n = -2p\hat{x} \quad (11)$$

the integral in (8) can be written as

$$\int_0^{\hat{x}} 1/\overset{0}{e}_x \, dx = \int_0^{\hat{x}} [m - px(2\hat{x}-x)] \, dx \quad (12)$$

for the interval 0 to $\hat{x}$. Since $\overset{0}{e}_0$ and $\overset{0}{e}_{\hat{x}}$ are known, the parameters can be solved from (10), and (12) can be evaluated. The same procedure may be applied to the interval $(\hat{x},5)$.

The equations (8) to (12) have been examined in some detail to verify the utility of this approach. For that, the model life table for males corresponding to level 70 ($\overset{0}{e}_0 = 53.6$) has been selected and the results shown in Table 3. This table has a life expectancy (combined for the two sexes) of 55 years which seems to be quite close to the value of that index in India at the present time.

Table 3. Graduated values of l_x compared with model values for model life tables level number 70 for males.

Age x (1)	$\overset{0}{e}_x$ (2)	1000 T_x Graduated (3)	1000 l_x Graduated (4)	1000 l_x Model (5)
0	53.6	53,565	1,000	1,000
1	58.8	52,644	895	882
5	56.4	49,186	872	840
10	53.9	44,920	833	828
15	50.7	40,821	805	819
20	47.0	36,844	787	806
25	42.9	32,958	768	787
30	38.5	29,138	757	767
35	34.0	25,370	746	748
40	29.7	21,668	730	726
45	25.4	18,052	711	699
50	21.5	14,564	677	665
55	17.8	11,266	633	620
60	14.6	8,249	564	561
65	11.8	5,624	477	484
70	9.4	3,488	371	387
75	7.3	1,898	260	274
80	5.6	862	154	162

It may be pointed out that the life expectancies were computed by rounding off at the first decimal digit and accordingly there is no sense in carrying out the computations of l_x with a radix of 100,000. A radix of 1,000 has therefore been chosen for which the two sets of figures demonstrate considerable closeness. It seems certain that the figures would be a lot closer if the computations of life expectancies were carried out to a few more significant digits and the numerical integrations were based on intervals shorter than five years.

4. Summary and Discussion

The model life tables prepared by the United Nations were based on a study that showed that the shape of the mortality curve is retained at all mortality levels and the infant mortality rate alone is generally sufficient for generating the entire mortality curve. The study reported here is based on the finding that the life expectancy can be regarded as an exponential function of age and for all practical purposes the logarithm of the former can be approximated by a quadratic equation of the latter variable namely, age, for the entire range except the childhood interval of less than 5 years. This age interval (0-4) also includes the age at which expectation of life assumes its maximum value and that age approaches the age 0 with increase in life expectancy. The model life tables were used to determine this age, the maximum life expectancy and the parameters of the quadratic equation for a number of levels, and the results were quite encouraging. Finally, it has been shown that a set of life expectancies is theoretically sufficient to generate the entire life table. The conclusion can therefore be drawn that while the expectation of life at birth, from the point at view of its definition uses the entire information of the life table, it can also be manipulated, under certain empirical conditions, to release all the information that it used with virtually little or no loss in that process.

Footnotes

1. Thus the force of mortality $\mu_x = Bc^x$

2. Makeham, W. M. "On the Law of Mortality and Construction of Annuity Tables" Journal of the Institute of Actuaries, Vol. 8, p. 301 (1860) Makeham wrote $\mu_x = A + Bc^x$ which by means of integration can also be written as $l_x = ks^x g^{c^x}$

3. Perks, W. "On Some Experiments in the Graduation of Mortality Statistics" Journal of the Institute of Actuaries, Vol. 63, p. 12 (1932) Perks modified the Makeham-Gompertz law by writing $\mu_x = (A + Bc^x)/(1 + Dc^x)$

4. Hyrenius, Hannes. "Life Table Technique for the Working Ages" Demography, Vol. 7, p. 393. (1970). The assumption is $m_x = a_0 + a_1 x + a_2 x^2 + \ldots$

5. Reed, L. J. and Merrell, M. "A Short Method for Constructing an Abridged Life Table" American Journal of Hygiene, Vol. 30, p. 33 (1939) The formula for age interval of length n years, is

$$_nq_x = 1 - e^{-n\,{}_nm_x - an^3\,{}_nm_x^2}$$

6. Greville, T. N. E., "Some Methods of Constructing Abridged Life Tables" The Record of the American Institute of Actuaries, Vol. 32, p. 29 (1943)

7. United Nations "Manuals on Methods of Estimating Population - Manual III, Methods for Population Projections by Sex and Age", ST/SOA/Series A, Population Studies, No. 25 (1956)

8. Coale, A. J. and Demeny P., Regional Model Life Tables and Stable Populations, Princeton University Press, Princeton, New Jersey (1966)

9. Mitra, S. "A Few properties of the Expectation of Life $\mathring{e}_x$" Presented at the Second World Population Conference in Belgrade, 1965

Appendix 1. Values of $\mathring{e}_x$ computed from model life table of males compared with those obtained by fitting a second degree curve to log $\mathring{e}_x$.

Note: M=Model, G=Graduated

Values of $\mathring{e}_x$ corresponding to model number

Age	10		25		40		55		70		85		100	
x	M	G	M	G	M	G	M	G	M	G	M	G	M	G
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
0*	24.8	24.8	31.9	31.9	39.2	39.2	46.4	46.4	53.6	53.6	61.5	61.5	68.5	68.5
5	38.1	37.6	43.7	42.5	49.1	47.5	54.0	52.7	58.6	56.4	62.6	60.4	66.3	64.3
10	35.8	35.1	40.7	40.1	45.7	45.1	50.2	50.0	54.4	53.9	58.2	57.8	61.6	61.3
15	32.3	32.4	37.0	37.4	41.6	42.3	45.9	46.8	49.9	50.7	53.5	54.3	56.8	57.6
20	29.1	29.6	33.5	34.4	37.9	39.0	42.0	43.2	45.7	47.0	49.1	50.3	52.1	53.3
25	26.2	26.7	30.5	31.2	34.6	35.5	38.4	39.3	41.8	42.9	44.8	45.9	47.5	48.5
30	23.4	23.8	27.4	27.9	31.2	31.8	34.7	35.2	37.8	38.5	40.5	41.2	42.8	43.5
35	20.7	21.0	24.4	24.6	27.8	28.1	31.0	31.0	33.7	34.0	36.1	36.5	38.2	38.6
40	18.1	18.3	21.4	21.5	24.5	24.5	27.2	27.0	29.7	29.7	31.8	31.8	33.6	33.5
45	15.8	15.8	18.6	18.4	21.2	21.0	23.6	23.1	25.7	25.4	27.5	27.2	29.1	28.8
50	13.6	13.4	15.9	15.6	18.1	17.8	20.1	19.5	21.9	21.5	23.4	23.0	24.8	24.3
55	11.6	11.2	13.4	13.0	15.2	14.8	16.8	16.2	18.3	17.8	19.6	19.1	20.8	20.2
60	9.7	9.4	11.0	10.8	12.4	12.1	13.7	13.3	15.0	14.6	16.0	15.7	17.0	16.6
65	7.9	7.7	8.9	8.8	9.9	9.8	11.0	10.8	11.9	11.8	12.8	12.6	13.6	13.4
70	6.2	6.3	7.0	7.0	7.8	7.8	8.5	8.6	9.3	9.4	10.0	10.0	10.6	10.6
75	4.8	5.0	5.4	5.6	6.0	6.2	6.5	6.7	7.1	7.3	7.6	7.8	8.0	8.4
80	3.6	4.0	4.0	4.3	4.5	4.8	4.9	5.2	5.3	5.6	5.6	6.0	5.9	6.4

(*Graduated $\mathring{e}_0$ has been assumed to be the same as model $\mathring{e}_0$)